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THE ENDOMORPHISM RING OF A Δ-MODULE OVER A RIGHT NOETHERIAN RING

BY

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ABSTRACT

Let R be a right noetherian ring. A module M_R is called a Δ-*module* provided R satisfies the descending chain condition for annihilators of subsets of M. For a Δ-module, a series $0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ can be constructed in which the factors M_i/M_{i-1} are sums of α_i-semicritical modules where $\alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_n$. In this paper we utilize this series in studying $\Lambda = \mathrm{End}(M_R)$. It is shown that if $N = \{f \in \Lambda \mid \mathrm{Ker}\, f$ is essential in $M\}$, then N is nilpotent. Specific bounds on the index of nilpotency are given in terms of this series. Further if M is injective and α-smooth, the annihilators of the factors of this series are used to provide necessary and sufficient conditions for $\mathrm{End}\, M_R$ to be semisimple.

1. Introduction

Throughout this paper, R denotes a right noetherian ring with Krull dimension α, i.e. $|R| = \alpha$. As in [7], a module M_R is termed a Δ-*module* provided R satisfies the descending chain condition for annihilators of subsets of M. In [14], it is shown that if M_R is a β-smooth module with $|R/\mathrm{ann}\, M| = \beta$, then M is a Δ-module. We show that every Δ-module over R has a close relationship to modules of this type. Given any Δ-module M, there exists a chain of submodules $0 \subset M_1 \subset \cdots \subset M_n = M$ such that the factors are β_i-smooth Δ-modules and $|R/\mathrm{ann}(M_i/M_{i-1})| = \beta_i$.

If $\Lambda = \mathrm{End}_R(M)$ and $N(\Lambda) = \{f \in \Lambda \mid \mathrm{Ker}\, f \leqq_e M\}$, then Shock provides conditions in [16] which assure that nil subrings of Λ are nilpotent. If the module M is noetherian, these conditions are satisfied. In [5], this result is extended to certain essential extensions of M. We show that if M is a finite dimensional Δ-module, then nil subrings of Λ are nilpotent. From this result it follows that if M is any α-smooth, finite dimensional R-module and if N is any essential extension of M, then in $\mathrm{End}_R N$ nil subrings are nilpotent.

Let M be a finite dimensional α-smooth module. In section 3, necessary and sufficient conditions for $\mathrm{End}_R(E(M))$ to be semisimple are provided in terms of

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linkage of α-coprimitive ideals of R. In this process, we determine exactly when $\mathrm{End}_R I$ is a division ring, where I is an α-indecomposable injective module.

In section 4, we apply the results of [7] and [14], to determine that the localization $R_{\mathcal{M}_\alpha}$ of R with respect to the topology of right ideals $\mathcal{M}_\alpha = \{H \leqq R_R \mid |R/H| < \alpha\}$ is semiprimary. Utilizing the results of section 3, we show that $R_{\mathcal{M}_\alpha}$ is semisimple if and only if $SI = I$ for all α-indecomposable injective modules I.

Throughout this paper, A denotes an arbitrary ring and R denotes a right noetherian ring with Krull dimension α.

From [3], [4] and [8], we have the following definitions and conventions. All modules are right unital. If S is a subset of M_A, then $S^r = \mathrm{ann}_A S = \{a \in A \mid sa = 0 \text{ for all } s \in S\}$, and if S is a subset of A then $S^l = \{x \in M \mid xs = 0$ for all $s \in S\}$. If N is an essential submodule of M, we write $N \leqq_e M$, while $E_A(M)$ or $E(M)$ denotes the injective hull of M_A.

Let β be an ordinal $\leqq \alpha$. An indecomposable injective module is called a β-indecomposable injective if it contains a β-critical module. The annihilator of a β-critical R-module is termed a *β-coprimitive* ideal. An R-module M is called *β-semicritical* provided there exists a finite collection of submodules $K_1, \cdots, K_n$ such that $\bigcap_{i=1}^n K_i = 0$, where M/K_i are β-critical for each i.

Let $\mathcal{M}_\beta$ be the topology [17] of right ideals H of R such that $|R/H| < \beta$, where β is an ordinal $-1 < \beta \leqq \alpha$. If N_R is a submodule of M, then the *β-closure* of N in M is $\mathrm{Cl}_\beta(N) = \{x \in M \mid xH \subseteq M$, for some $H \in \mathcal{M}_\beta\}$. Now M is *β-torsionfree* provided $\mathrm{Cl}_\beta(0) = 0$, which implies that $|xR| \geqq \beta$ for all $x \in M$. If M is β-torsionfree, and there exists a nonzero $x \in M$ such that $|xR| = \beta$, then M is termed a *β-module*. We will say that a module M_R is *β-smooth* provided for every finitely generated submodule N of M, we have $|N| = \beta$. This is equivalent to the statement that every submodule N of M, which has Krull dimension, has Krull dimension β.

2. The semicritical socle series of Δ-modules and their endomorphism ring

Let M be a right R-module. If β is an ordinal, $\beta \leqq \alpha = |R|$, then the *β-semicritical socle* of M is defined to be the sum of the β-semicritical submodules of M. This is denoted $\mathrm{Sc}^\beta(M)$. If M has no β-semicritical modules, then $\mathrm{Sc}^\beta(M) = 0$. The *β-critical socle* of M is the sum of the β-critical submodules of M and is denoted $S^\beta(M)$. Clearly $S^\beta(M) \subset \mathrm{Sc}^\beta(M)$.

If M is β-torsionfree, it is shown in [4] that the β-closure, $\mathrm{Cl}_\beta(S^\beta M)$, of $S^\beta M$ in M is $\mathrm{Sc}^\beta M$ and hence $M/\mathrm{Sc}^\beta M$ is β-torsionfree. This enables us to define the following series. The β-semicritical socle series for M is defined inductively: $\mathrm{Sc}_1^\beta M = \mathrm{Sc}^\beta M$, $\overline{\mathrm{Sc}_{n+1}^\beta M} = \mathrm{Sc}^\beta(M/\mathrm{Sc}_n^\beta M)$. Thus $\mathrm{Sc}_1^\beta M \subset \mathrm{Sc}_2^\beta M \subset \cdots \subset M$, and this is called the *β-semicritical* socle series for M. For simplicity, if M is β-torsionfree, we will delete the β-superscript when referring to the β-semicritical socle series for M. Thus we write $\mathrm{Sc}_n M$ for $\mathrm{Sc}_n^\beta M$.

Let M be a right R module and let β_1 denote the minimal Krull dimension among nonzero submodules of M. In general the β_1-semicritical socle series for M need not be finite. However in the case when this is finite, i.e. $\bigcup_{i=1}^\infty \mathrm{Sc}_i^{\beta_1} M = \mathrm{Sc}_{n(1)}^{\beta_1} M$ for some integer $n(1)$, we can continue this procedure. Let $M_1 = \mathrm{Sc}_{n(1)}^{\beta_1} M$. Since the β_1-semicritical socle series has stopped at M_1 necessarily M/M_1 has no submodules of Krull dimension β_1. Let $\beta_2 > \beta_1$ denote the minimal Krull dimension among nonzero submodules of M/M_1. Then M/M_1 is β_2-torsionfree and we can construct a β_2-semicritical socle series for M/M_2. If at each stage we obtain a finite β_i-semicritical socle series for M/M_{i-1}, and if there are only a finite number of these stages, we will say that M has a *finite semicritical socle series* and write

$$\begin{aligned} 0 \subset \mathrm{Sc}_1^{\beta_1} M \subset \cdots \subset \mathrm{Sc}_{n(1)}^{\beta_1} M &= M_1 \subset \mathrm{Sc}_1^{\beta_2} M \subset \cdots \subset \mathrm{Sc}_{n(2)}^{\beta_2} M \\ (*) \qquad &= M_2 \subset \cdots \subset M_{k-1} \subset \mathrm{Sc}_1^{\beta_k} M \subset \cdots \subset \mathrm{Sc}_{n(k)}^{\beta_k} M \\ &= M_k = M. \end{aligned}$$

By construction, M/M_{i-1} is β_i-torsionfree and M_i/M_{i-1} is β_i-smooth. In addition, $\beta_i > \beta_j$ for $i > j$.

2.1. Lemma. *Let M be a module with finite semicritical socle series* (*). *Let N be a nonzero submodule of M which has Krull dimension. Then $N \subset M_i$ iff $|N| \leqq \beta_i$. Furthermore, $|N| = \beta_j$ for some j, $1 \leqq j \leqq k$.*

Proof. Suppose $|N| \leqq \beta_i$. Since M/M_i is β_{i+1}-torsionfree and since $|N + M_i/M_i| \leqq |N| \leqq \beta_i < \beta_{i+1}$, then $N + M_i/M_i \subseteq M/M_i$ implies that $N + M_i/M_i = 0$, and $N \subseteq M_i$.

The remainder is proved inductively. Since M_1 is β_1-smooth, if $N \subset M_1$, then $|N| = \beta_1$. Suppose $N \subset M_i$. Let j denote the smallest integer such that $N \subset M_j$. Then $N + M_{j-1}/M_{j-1}$ is a nonzero submodule of the β_j-smooth module M_j/M_{j-1} and thus $|N + M_{j-1}/M_{j-1}| = \beta_j$. By induction $|N \cap M_{j-1}| \leqq \beta_{j-1}$. Thus $|N| = \sup\{|N + M_{j-1}/M_{j-1}|, |N \cap M_{j-1}|\} = \beta_j \leqq \beta_i$.

Recall that an A module M is called a *Δ-module* if A satisfies the descending chain condition on annihilators of subsets of M. In the next sequence of results we show that every Δ-module over the right noetherian ring R has a finite semicritical socle series.

Over a ring with Krull dimension every smooth module having the same Krull dimension as the ring, is a Δ-module. This is verified in [14, 2.3] and is stated below for reference.

2.2. LEMMA. *Let A be a ring with Krull dimension β. If M is a β-smooth module, then M is a Δ-module.*

Thus for the ring R, where $|R| = \alpha$, every α-module is a Δ-module. The next result provides us with a partial converse — if M is a β-smooth Δ-module then $|R/\operatorname{ann} M| = \beta$. This enables us to utilize properties of the α-semicritical socle series in [4].

If M is a right R-module, let $\bar{M}$ denote the quasi-injective hull of M. Unlike $E(M)$, when M is β-smooth the same is true of $\bar{M}$ since $\bar{M} = \Lambda M$, where $\Lambda = \operatorname{End}_R(E(M))$. This close relationship between M and $\bar{M}$ enables us to prove the following.

A module M_R is *finitely annihilated* if there exists $x_1, \cdots, x_n \in M$ such that $\operatorname{ann} M = x_1^r \cap \cdots \cap x_n^r$.

2.3. THEOREM. *Let M_R be a β-smooth module with quasi-injective hull $\bar{M}$. The following are equivalent.*

(1) *$R/\operatorname{ann} M$ is β-smooth.*

(2) *M is a Δ-module.*

(3) *M is finitely annihilated.*

(4) *There exists a ring S such that $\bar{M}$ is an (S, R)-bimodule and ${}_S\bar{M}$ is finitely generated.*

In this case M and $\bar{M}$ have finite β-semicritical socle series. Furthermore the ring S can be chosen to be $\operatorname{End}_R \bar{M}$.

PROOF. (1)→(2): Since $\beta = |M| = |R/\operatorname{ann} M|$, this follows from 2.2.

(2)→(3): Clear.

(3)→(4): Let $S = \operatorname{End}_R \bar{M}$. By [1, 1.5], ${}_S\bar{M}$ is finitely generated.

(4)→(1): Since ${}_S\bar{M}$ is finitely generated, there exists $x_1, \cdots, x_n \in \bar{M}$ such that $Sx_1 + \cdots + Sx_n = \bar{M}$. Thus $\operatorname{ann}(\bar{M}) = \bigcap_{i=1}^n \operatorname{ann}(x_i)$. By [1, 1.5], there exist $m_1, \cdots, m_t \in M$ such that $\operatorname{ann} M = \bigcap_{i=1}^t \operatorname{ann}(m_i)$. This provides a monomorphism $R/\operatorname{ann} M \to M^{(t)}$. Since M is β-smooth necessarily $R/\operatorname{ann} M$ is β-smooth.

Since $|R/\operatorname{ann} M| = \beta = |M|$, then M has a finite semicritical socle series by [4, 3.3]. Similarly this is true for $\bar{M}$.

The equivalence of (3) and (4) can also be found in [7, p. 15].

Neither the injective hull nor the quasi-injective hull of a Δ-module is necessarily a Δ-module as can be seen in the following example. Let $R = \mathbf{Z}$ and $M = Q \oplus \mathbf{Z}_p$. Then M is a Δ-module. However $\bar{M} = E(M) = Q \oplus \mathbf{Z}_p^\infty$ which is not a Δ-module. Note that M is not smooth. If M is β-smooth for some β then the proof of 2.3 shows that $\bar{M}$ is also a Δ-module. As a result, 2.3 can also be established by means of the results in section 4.

2.4. THEOREM. *If M_R is a Δ-module, then M has a finite semicritical socle series.*

PROOF. Let β_1 denote the minimal Krull dimension of nonzero submodules of M and let $M_1 = \bigcup_{i=1}^{\infty} \mathrm{Sc}_i^{\beta_1} M$. Then M_1 is β_1-smooth and is a Δ-module. By 2.3, $|R/\mathrm{ann}\, M_1| = \beta_1$. Hence by [4, 3.3], M_1 has a finite semicritical socle series.

Let $\beta_2 > \beta_1$ denote the minimal Krull dimension of nonzero submodules of M/M_1, and let $M_2/M_1 = \bigcup_{i=1}^{\infty} \mathrm{Sc}_i^{\beta_2}(M/M_1)$. Then M_2/M_1 is a β_2-smooth module. Since M_2 is a Δ-module, there exist $x_1, \cdots, x_n \in M_2$ such that $\mathrm{ann}\, M_2 = \bigcap_{i=1}^{n} \mathrm{ann}(x_i)$. Then either $x_i \in M_1$ and $|x_i R| = \beta_1$ or $x_i \in M_2 - M_1$ and $|x_i R| = \beta_2$ by 2.1. Thus $|R/\mathrm{ann}(x_i)| = \beta_1$ or β_2 and since $|R/\mathrm{ann}\, M_2| = \sup_{1 \leq i \leq n}\{|R/\mathrm{ann}(x_i)|\}$, then $|R/\mathrm{ann}\, M_2| = \beta_1$ or β_2. However M_2/M_1 is an $R/\mathrm{ann}\, M_2$ module which is β_2-smooth and hence $|R/\mathrm{ann}\, M_2| = \beta_2$. By [4, 3.3], since M_2/M_1 is β_2-smooth and $|R/\mathrm{ann}\, M_2| = \beta_2$, M_2/M_1 has a finite β_2-semicritical, socle series.

Continuing in this fashion we generate a sequence of modules $M_1 \subset M_2 \subset M_3 \subset \cdots \subset M$ where M_{i+1}/M_i is β_{i+1}-smooth and $|R/\mathrm{ann}\, M_{i+1}| = \beta_{i+1}$. The sequence $\{\mathrm{ann}_R M_i\}$ is a descending chain which must stop, since M is a Δ-module. Thus there exists an integer n such that $\mathrm{ann}_R M_n = \mathrm{ann}_R M_j$ for all $j \geqq n$. This implies that $\beta_n = \beta_j$ for all $j \geqq n$ and hence that $M_n = M_j$ for all $j \geqq n$. Thus $M_n = M$. Since each M_{i+1}/M_i has a finite semicritical socle series, the same is true of M.

From the above proof we have that M_i/M_{i-1} is β_i-smooth and that $|R/\mathrm{ann}\, M_i| = \beta_i$ which implies that $|R/\mathrm{ann}(M_i/M_{i-1})| = \beta_i$. Thus from 2.3, M_i/M_{i-1} is a β_i-smooth Δ-module. Hence 2.4 yields a chain $0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ in which the factors are smooth Δ-modules over R.

We now examine the endomorphism ring Λ of a Δ-module in light of the structure provided by 2.4. We show that if M is finite dimensional then nil subrings of Λ are nilpotent without assuming any additional conditions on M, such as the condition that M is injective. Our procedure utilizes $N(\Lambda) = \{f \in \Lambda \mid \mathrm{Ker}\, f \leqq_e M\}$. We begin by examining the interaction of $N(\Lambda)$ and the semicritical socle series. Recall if M is β-torsionfree, then we write $\mathrm{Sc}_i M$ in place of $\mathrm{Sc}_i^{\beta} M$.

2.5. LEMMA. *Let M and N be right R-modules and let $f \in \mathrm{Hom}_R(M,N)$.*

(1) *If* $\mathrm{Ker} f \leqq_e M$ *and* $x \in \mathrm{Sc}^\beta M$, *then* $|f(xR)| < \beta$.

(2) *If M and N are β-torsionfree, then* $f(\mathrm{Sc}_i M) \subset \mathrm{Sc}_i N$ *for all i.*

(3) *If M and N are β-torsionfree and* $\mathrm{Ker} f \leqq_e M$, *then* $f(\mathrm{Sc}_1 M) = 0$.

PROOF. (1) Let $x \in \mathrm{Sc}^\beta M$. Then $x = x_1 + \cdots + x_n$ where $x_i R$ is β-semicritical and $xR \subseteq \Sigma_{i=1}^n x_i R$. Thus $|f(xR)| \leqq \sup_{1 \leqq i \leqq n} |f(x_i R)|$. Since $x_i R$ is β-semicritical and $\mathrm{Ker} f \cap x_i R \leqq_e x_i R$, by [3, 2.4], $|f(x_i R)| = |x_i R / x_i R \cap \mathrm{Ker} f| < \beta$. Hence $|f(xR)| < \beta$.

(2) Let C be a β-critical submodule of M. If $\mathrm{Ker} f \cap C \neq 0$, then $f(C) = 0$ since N is β-torsionfree; otherwise $f(C) \cong C$. Hence $f(S^\beta M) \subset S^\beta N$. If $x \in \mathrm{Sc}_1 M = \mathrm{Cl}_\beta(S^\beta M)$, then $xT \subset S^\beta M$ for some $T \in \mathcal{M}_\beta$. Hence $f(x)T = f(xT) \subseteq S^\beta N$ which implies that $f(x) \in \mathrm{Cl}_\beta(S^\beta N) = \mathrm{Sc}_1 N$.

Proceeding inductively, if $f(\mathrm{Sc}_i M) \subseteq \mathrm{Sc}_i N$, then f induces a mapping $f^*: M/\mathrm{Sc}_i M \to N/\mathrm{Sc}_i N$ where $f^*(m + \mathrm{Sc}_i M) = f(m) + \mathrm{Sc}_i N$. Then

$$f^*(\mathrm{Sc}_{i+1} M/\mathrm{Sc}_i M) = f(\mathrm{Sc}^\beta(M/\mathrm{Sc}_i M)) \subseteq \mathrm{Sc}^\beta(N/\mathrm{Sc}_i N) = \mathrm{Sc}_{i+1} N/\mathrm{Sc}_i N.$$

Thus $f(\mathrm{Sc}_{i+1} M) \subseteq \mathrm{Sc}_{i+1} N$.

(3) This follows from (1), since N is β-torsionfree.

2.6. PROPOSITION. *Let M be a β-torsionfree module with β-semicritical socle series* $0 \subset \mathrm{Sc}_1 M \subset \mathrm{Sc}_2 M \subset \cdots \subset M$. *If* $\Lambda = \mathrm{End}_R M$ *and* $N = N(\Lambda)$, *then* $N^k \cdot \mathrm{Sc}_k M = 0$ *for all k.*

If $\Lambda_k = \mathrm{End}_R(\mathrm{Sc}_k M)$ *and* $N_k = N(\Lambda_k)$, *then* $N_k^k = 0$.

PROOF. By 2.5, if $f \in N$, then $f(\mathrm{Sc}_1 M) = 0$. Thus $N \cdot \mathrm{Sc}_1 M = 0$. Furthermore, f induces a mapping of $M/\mathrm{Sc}_1 M$ into M. Since $\mathrm{Sc}_2 M/\mathrm{Sc}_1 M = \mathrm{Sc}_1(M/\mathrm{Sc}_1 M)$, by 2.5, $f(\mathrm{Sc}_2 M) \subseteq \mathrm{Sc}_1 M$. Hence $N \cdot f(\mathrm{Sc}_2 M) = 0$ for all $f \in N$ and therefore $N^2 \mathrm{Sc}_2 M = 0$. Continuing in this fashion we have that $N^k \cdot \mathrm{Sc}_k M = 0$. The proof of the last statement is similar.

Recall that a ring R is termed *semiperfect* provided that idempotents modulo the Jacobson radical $J(R)$ can be lifted, and $R/J(R)$ is semisimple artinian. A semiperfect ring is *semiprimary* if $J(R)$ is nilpotent.

If M is a Δ-module over R with endomorphism ring Λ, we now show that $N(\Lambda)$ is nilpotent. Thus if M is finite dimensional and quasi-injective then Λ is semiprimary.

2.7. THEOREM. *Let M be a Δ-module with finite semicritical socle series* (*). *Let* $\Lambda = \mathrm{End}_R M$ *and* $N = N(\Lambda) = \{f \in \Lambda \mid \mathrm{Ker} f \leqq_e M\}$. *Then N is nilpotent.*

Furthermore, the index of nilpotency of N is $\leqq$ the sum of the nonleading coefficients of the polynomial $\Pi_{i=1}^{k}(x+n(i))$.

PROOF. We first verify the following. If $N^pM_i = 0$, then $N^{jp+j+p}\mathrm{Sc}_j^{\beta_{i+1}}M = 0$. The proof is by induction on j, where $0 \leqq j \leqq n(i)$. If $j = 0$, then $\mathrm{Sc}_0^{\beta_{i+1}}M = M_i$ and hence $0 = N^pM_i = N^p\mathrm{Sc}_0^{\beta_{i+1}}M$. Assume $j \geqq 0$ and let $f \in N^{jp+p+j}$. By induction, $f(\mathrm{Sc}_j^{\beta_{i+1}}M) = 0$. Let $x \in \mathrm{Sc}_{j+1}^{\beta_{i+1}}M$. Then $f(xR)$ is a homomorphic image of the β_{i+1}-semicritical module $\overline{xR} = xR + \mathrm{Sc}_j^{\beta_{i+1}}M/\mathrm{Sc}_j^{\beta_{i+1}}M$. Thus if

$$\overline{\mathrm{Ker}\, f} = (\mathrm{Ker}\, f \cap xR) + \mathrm{Sc}_j^{\beta_{i+1}}M/\mathrm{Sc}_j^{\beta_{i+1}}M \leqq_e \overline{xR},$$

then $|f(xR)| < \beta_{i+1}$ by [3, 2.4] and hence $f(xR) \subset M_i$ by 2.1.

If $\overline{\mathrm{Ker}\, f}$ is not essential in $\overline{xR}$, let $\bar{D}$ denote its relative complement in $\overline{xR}$. Since $f(\mathrm{Sc}_j^{\beta_{i+1}}M) = 0$, f induces a map $f^* : \overline{xR} \to M$, where $f^*(\overline{xr}) = f(xr)$. Then $f^*(\bar{D} + \overline{\mathrm{Ker}\, f}) \cong \bar{D}$ and hence $f^*(\bar{D} + \overline{\mathrm{Ker}\, f}) \subset \mathrm{Sc}^{\beta_{i+1}}M$. Let $h \in N$. By 2.5 and 2.1, $h(\mathrm{Sc}^{\beta_{i+1}}M) \subset M_i$. Thus $hf^*(\overline{\bar{D} + \mathrm{Ker}\, f}) \subset M_i$. Since $\overline{xR}$ is β_{i+1}-semicritical and $\overline{\mathrm{Ker}\, f} + \bar{D} \leqq_e \overline{xR}$, then $|\overline{xR}/\overline{\mathrm{Ker}\, f} + \bar{D}| < \beta_{i+1}$ by [3, 2.4] and hence

$$|hf^*(\overline{xR})/hf^*(\overline{\mathrm{Ker}\, f} + \bar{D})| < \beta_{i+1}.$$

Since $hf^*(\overline{\mathrm{Ker}\, f} + \bar{D}) \subset M_i$, then $|hf^*(\overline{xR}) + M_i/M_i| < \beta_{i+1}$. However M/M_i is β_{i+1}-torsionfree which necessitates that $hf^*(\overline{xR}) = hf(xR) \subset M_i$.

Thus in either case $hf(xR) \subset M_i$. By hypothesis, $N^phf(xR) = 0$. Since f, h and x were chosen arbitrarily, we have that $0 = N^pNN^{jp+p+j}\mathrm{Sc}_{j+1}^{\beta_{i+1}}M = N^{(j+1)p+p+(j+1)}\mathrm{Sc}_{j+1}^{\beta_{i+1}}M = 0$, which verifies the claim.

By 2.6, $N^{n(1)}M_1 = 0$. Since $M_2 = \mathrm{Sc}_{n(2)}^{\beta_2}M$, then from the above argument we have that $N^{n(1)n(2)+n(1)+n(2)}M_2 = 0$. Note that $n(1)n(2) + n(1) + n(2)$ is the sum of the nonleading coefficients of $(x + n(1))(x + n(2))$. Continuing in this manner we obtain the desired result that the bound on the index of nilpotency of N is the sum of the elementary symmetric polynomials in the symbols $n(1), \cdots, n(k)$.

The bound on the index of nilpotency in 2.7 cannot be improved as the following example shows. Let

$$R = M = \begin{bmatrix} \mathbf{Z} & \mathbf{Z}_p \\ 0 & \mathbf{Z}_p \end{bmatrix} \oplus \mathbf{Z}.$$

The semicritical socle series (*) for M is

$$0 \subset \begin{bmatrix} 0 & \mathbf{Z}_p \\ 0 & \mathbf{Z}_p \end{bmatrix} = M_1 \subset \begin{bmatrix} \mathbf{Z} & \mathbf{Z}_p \\ 0 & \mathbf{Z}_p \end{bmatrix} \oplus \mathbf{Z} = M_2 = M.$$

Hence $n(1) = n(2) = 1$. Let f, $h \in \Lambda$ where

$$f\left(\begin{bmatrix} a & \bar{b} \\ 0 & \bar{c} \end{bmatrix}, n\right) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, a\right) \quad \text{and} \quad h\left(\begin{bmatrix} a & \bar{b} \\ 0 & \bar{c} \end{bmatrix}, n\right) = \left(\begin{bmatrix} 0 & 0 \\ 0 & \bar{n} \end{bmatrix}, 0\right).$$

Then f, $h \in N(\Lambda)$ and $h \cdot f \neq 0$. Thus $N^2 \neq 0$. The index of nilpotency of N is $3 = n(1)n(2) + n(1) + n(2)$.

2.8. Corollary. *Let M be a Δ-module with semicritical socle series* (*). *Let $\Lambda = \mathrm{End}_R M$ and $N = N(\Lambda)$. If $S^{\beta_1}M \leqq_e M$, then the index of nilpotency of N is $\leqq n(1)n(2)\cdots n(k)$.*

Proof. If $S^{\beta_1}M \leqq_e M$, then $\mathrm{Sc}^{\beta_i}N = 0$ for all $2 \leqq i \leqq k$. The proof now follows as in 2.7 with this modification.

Combining 2.7 and [16, 3] we have

2.9. Corollary. *If M is a finite dimensional Δ-module, then nil subrings of $\mathrm{End}_R M$ are nilpotent.*

Every essential extension of an α-smooth R module is α-smooth and therefore is a Δ-module by 2.2. Thus

2.10. Corollary. *If M is a finitely general α-module over R and T is any essential extension of M, then nil subrings of* $\mathrm{End}_R T$ are nilpotent.

If H is a semiperfect ring with Jacobson radical $J(H)$, and T an ideal of H such that $J(H)^n \subseteq T \subsetneq J(H)$, then H/T is semiprimary. This is precisely the relationship which exists between $\mathrm{End}(M)$ and $\mathrm{End}(\mathrm{Sc}_i M)$ when M is a finite dimensional β-smooth injective module. In this situation we are not assuming that M necessarily has a finite semicritical socle series.

2.11. Proposition. *Let M be a β-smooth, finite dimensional injective module over R with semicritical socle series $0 = \mathrm{Sc}_0 M \subset \mathrm{Sc}_1 M \subset \cdots \subset M$. Let $H = \mathrm{End}_R M$, $J = J(H)$, $H_i = \mathrm{End}_R(\mathrm{Sc}_i M)$ and $J_i = J(H_i)$. Then*

(1) *H is semiperfect and H_i is semiprimary for all i.*

(2) *$H_i \cong H/T_i$ where T_i is an ideal of H such that $J^i \subseteq T_i \subseteq J$, $T_i \subseteq T_j$ for $j \leqq i$, and $\bigcap_{i=1}^{\infty} T_i = 0$.*

(3) *$H_i/J_i \cong H/J$.*

Proof. (1) Since M is a finite dimensional injective, and $\mathrm{Sc}_i M$ is quasi-injective by 2.5, we have $J(H) = N(H)$ and $J(H_i) = N(H_i)$. Now apply 2.6.

(2) By 2.5, $\mathrm{Sc}_i M$ is quasi-injective for all i. Thus the mapping $\theta_i : H \to H_i$ where $\theta_i(h) = h|_{\mathrm{Sc}_i M}$ is a ring homomorphism which is surjective. Let $T_i = \mathrm{Ker}\,\theta_i$. By 2.6, $J_i^i = 0$ and hence $J^i \subseteq T_i$ since $H_i \cong H/T_i$. If $h \in T_i$ then

$\mathrm{Sc}_i M \subseteq \mathrm{Ker}\, h$. Since M is β-smooth, then $\mathrm{Sc}_i M \leqq_e M$, and thus $\mathrm{Ker}\, h \leqq_e M$. Hence $T_i \subset J$. Clearly $T_i \subseteq T_j$ for $j \leqq i$, and $\bigcap_{i=1}^{\infty} T_i = 0$, since $M = \bigcup_{i=1}^{\infty} \mathrm{Sc}_i M$ and $(\bigcap_{i=1}^{\infty} T_i)(M) = 0$.

(3) This is clear since $\theta_i(J) = J_i$.

Let I be an α-smooth indecomposable injective module. By [8, 2.2], ann(Sc I) is a minimal α-coprimitive D. Furthermore, by 2.11, $\mathrm{End}(\mathrm{Sc}\, I) \cong H/J(H)$ is a division ring F. Then using [4, 4.11] and chapter 2 of [11], one can show the following.

2.12. Proposition. *Let I be an α-indecomposable injective module, let $D =$ $\mathrm{ann}(\mathrm{Sc}^{\alpha} I)$. Then $F = \mathrm{End}_R \mathrm{Sc}^{\alpha} I$ is a division ring. Furthermore $\mathrm{Sc}^{\alpha} I$ is a finite dimensional vector space over F with dimension equal to the uniform dimension of R/D.*

3. When End M is semisimple

Let M be an α-smooth injective module over R. In this section we examine the annihilators of the factors of the α-semicritical socle series of M, and provide necessary and sufficient conditions for $\mathrm{End}_R M$ to be semisimple.

From [8], there exists a one-one correspondence between the isomorphism classes of α-indecomposable injectives and minimal α-coprimitive ideals given by $I \to D = \mathrm{ann}(S^{\alpha} I)$. We will say that D is the *minimal α-coprimitive ideal associated with I.* Using this relationship, it is shown in [4, 2.11] that $\mathrm{ann}(\mathrm{Sc}_i^{\alpha} M/\mathrm{Sc}_{i-1}^{\alpha} M)$ is a finite intersection of α-coprimitive ideals. Since the annihilators of α-smooth modules are determined, we will restrict our attention to this situation.

Let M_R be an α-smooth module with α-semicritical socle series $0 \subset \mathrm{Sc}_1 M \subset \mathrm{Sc}_2 M \subset \cdots \subset \mathrm{Sc}_n M = M$. A minimal α-coprimitive ideal D is *linked to M* if D annihilates a nonzero submodule of $\mathrm{Sc}_i M/\mathrm{Sc}_{i-1} M$ for some i, $1 \leqq i \leqq n$. In this case we say that D is *linked to M at the ith layer.* By [4, 4.4] this is equivalent to saying that D annihilates a factor module in a critical composition series for N, where N is a finitely generated submodule of M_R.

3.1. Proposition. *Let D be a minimal α-coprimitive ideal of R and let I denote the associated α-indecomposable injective. If M is an α-smooth module, then D is linked to the ith layer of M if and only if there exists a nonzero homomorphism of $\mathrm{Sc}_i M/\mathrm{Sc}_{i-1} M$ into I.*

Proof. If D is linked to the ith layer of M, then D annihilates a critical submodule $\bar{C} = C + \mathrm{Sc}_{i-1}M/\mathrm{Sc}_{i-1}M$ of $\mathrm{Sc}_i M/\mathrm{Sc}_{i-1}M$. Since $\mathrm{Sc}_i M/\mathrm{Sc}_{i-1}M$ is α-smooth, $|\bar{C}| = \alpha$. Hence by [8, 2.1], $E(\bar{C}) \cong I$. The composite of the projection map of C onto $\bar{C}$ and the inclusion of $\bar{C}$ into I provides a nonzero map $f: C \to \bar{C} \to I$. This extends to a map of $\mathrm{Sc}_i M/\mathrm{Sc}_{i-1}M$ into I by the injectivity of I.

Conversely, suppose $f: \mathrm{Sc}_i M/\mathrm{Sc}_{i-1}M \to I$ is a nonzero homomorphism. If $S^\alpha(\mathrm{Sc}_i M/\mathrm{Sc}_{i-1}M) \subseteq \mathrm{Ker} f$, then $\mathrm{Ker} f \leqq_e \mathrm{Sc}_i M/\mathrm{Sc}_{i-1}M$. By 2.5, we then have $f = 0$, which is a contradiction. Necessarily then there exists a critical submodule $\bar{C}$ of $\mathrm{Sc}_i M/\mathrm{Sc}_{i-1}M$ such that $f(\bar{C}) \neq 0$. Since the minimal Krull dimension of nonzero submodules of I is α, then $f(\bar{C}) \cong \bar{C}$. By [8, 2.2], $D = \mathrm{ann}\, S^\alpha I$ and therefore, $0 = f(\bar{C})D = f(\bar{C}D)$, which implies that $\bar{C}D = 0$. Thus D annihilates a nonzero submodule of $\mathrm{Sc}_i M/\mathrm{Sc}_{i-1}M$ and hence is linked to M at the ith layer.

If D and D^* are minimal α-coprimitive ideals of R and I^* is the α-indecomposable injective associated with D^* then we say that *D is linked to D^** if D is linked to I^*.

3.2. Corollary. *Let D and D^* be minimal α-coprimitive ideals of R with associated α-indecomposable injectives I and I^*, respectively. Then D is linked to D^* if and only if* $\mathrm{Hom}_R(I^*, I) \neq 0$.

Proof. If D is linked to D^*, then D annihilates a nonzero submodule of $\mathrm{Sc}_i I^*/\mathrm{Sc}_{i-1}I^*$ for some i. By 3.1, there exists a nonzero map $f: \mathrm{Sc}_i I^*/\mathrm{Sc}_{i-1}I^* \to I$. Thus $\pi \circ f: \mathrm{Sc}_i I^* \to \mathrm{Sc}_i I^*/\mathrm{Sc}_{i-1}I^* \to I$ is a nonzero homomorphism which extends to $f^*: I^* \to I$, by the injectivity of I.

Conversely, let $f: I^* \to I$ be a nonzero homomorphism. Since $I^* = \bigcup_{i=1}^n \mathrm{Sc}_i(I^*)$, there exists a smallest integer t such that $\mathrm{Sc}_t I^* \not\subseteq \mathrm{Ker} f$. Then f induces a nonzero homomorphism $\bar{f}: \mathrm{Sc}_t I^*/\mathrm{Sc}_{t-1}I^* \to I$. By 3.1, D is linked to I^*, and hence to D^*.

In [6], Faith provides conditions which determine when End I is a division ring for an arbitrary indecomposable injective. Linkage can be used to this purpose in the following way.

3.3. Theorem. *Let I_R be an α-indecomposable injective module with α-semicritical socle series $0 \subset \mathrm{Sc}_1 I \subset \mathrm{Sc}_2 I \subset \cdots \subset \mathrm{Sc}_n I = I$. Let D be the minimal α-coprimitive ideal associated with I. The following are equivalent.*

(1) $\Lambda = \mathrm{End}_R I$ *is a division ring.*

(2) $\mathrm{Hom}(I/K, I) = 0$ *for all nonzero submodules K of I.*

(3) $\mathrm{Hom}(I/\mathrm{Sc}_i I, I) = 0$ *for all i, $1 \leqq i \leqq n$.*

(4) *D is linked to I at only the first layer.*

PROOF. The proof of (1)→(2) and (2)→(3) is direct. Now (3)→(4) follows from 3.1.

(4)→(1): Let J denote the Jacobson radical of Λ. Then $J = \{j \in \Lambda \mid \operatorname{Ker} j \leqq_e I\}$ and it suffices to show that $J = 0$. If $0 \neq j \in J$, let i denote the smallest positive integer such that $j(\mathrm{Sc}_i I) \neq 0$. By 2.5, $j(\mathrm{Sc}_1 I) = 0$ and hence $i > 1$. Then j induces a nonzero map $j^* : \mathrm{Sc}_i I/\mathrm{Sc}_{i-1} I \to I$, which implies by 3.1 that D is linked to the ith layer of I, where $i > 1$. Thus $J = 0$, and $\mathrm{End}_R I$ is a division ring.

By 2.3, if I is a β-smooth Δ-module then $|R/\operatorname{ann} I| = \beta$. Thus an analogous statement to 3.3 can be made for an indecomposable injective which is a β-smooth Δ-module.

3.4. THEOREM. *Let M be a finite dimensional α-smooth module and let $E(M) = I_1 \oplus \cdots \oplus I_n$, where I_i is an α-indecomposable injective for $1 \leqq i \leqq n$. Let D_i denote the minimal α-coprimitive ideal associated with I_i. Then $\mathrm{End}_R E(M)$ is semisimple if and only if D_i is linked to I_i at only the first layer for $1 \leqq i \leqq n$, and D_i is not linked to D_j unless $I_i \cong I_j$.*

PROOF. By 3.2, if D_i is not linked to D_j unless $I_i \cong I_j$, then $\mathrm{Hom}(I_i, I_j) = 0$ for $I_i \not\cong I_j$. Further by 3.3, $\mathrm{End}_R I_i$ is a division ring. Hence $\mathrm{End}_R E(M)$ is semisimple. Conversely suppose $\mathrm{End}_R E(M)$ is semisimple. Then $J = \{j \in \mathrm{End}\, E(M) \mid \operatorname{Ker} j \leqq_e E(M)\} = 0$. By 3.1 if D_j is linked to I_j at the ith layer, there exists a nonzero map $f : I_j/\mathrm{Sc}_{i-1} I_j \to I_j$ and hence a nonzero map $f^* : I_j \to I_j/\mathrm{Sc}_{i-1} I_j \to I_j$ where $\operatorname{Ker} f^* \supseteq \mathrm{Sc}_{i-1} I$. Then f^* extends to a map $k : E(M) \to E(M)$ where $k([i_1, \cdots, i_n]) = f^*(i_j)$ for $i_t \in I_t$ where $1 \leqq t \leqq n$. Then $k \in J = 0$, which implies that $f = 0$ unless $i = 1$.

If $f : I_i \to I_j$ where $I_i \not\cong I_j$, then $\operatorname{Ker} f \leqq_e I_i$. Again f extends to a map $k : E(M) \to E(M)$ where $k([i_1, \cdots, i_n]) = f(i_i)$. Then $k \in J = 0$ which implies that $f = 0$. By 3.2, D_i is not linked to D_j.

From [8, 2.5] for the right noetherian ring R, there is only a finite number of α-coprimitive ideals, and hence only a finite number of isomorphism classes of α-indecomposable injectives. Let $I_1, \cdots, I_k$ be a complete set of representatives of these injectives. If R is an α-smooth ring, then in [4, 4.11] we show that R is semicritical provided $S^\alpha I_i = I_i$ for all i, $1 \leqq i \leqq k$. We get another equivalence using 3.4.

3.5. COROLLARY. *Let $I_1, \cdots, I_k$ denote a complete set of distinct representatives of the isomorphism classes of α-indecomposable injectives. Let $M = I_1 \oplus I_2 \oplus \cdots \oplus I_k$. Then $\mathrm{End}_R M$ is semisimple artinian if and only if $S^\alpha I_i = I_i$ for all $1 \leqq i \leqq k$.*

Let Q be a uniform quasi-injective module which is α-smooth and let $0 \subset \mathrm{Sc}_1 Q \subset \cdots \subset \mathrm{Sc}_n Q = Q$ be an α-semicritical socle series for Q. In 2.7 it is shown that the Jacobson radical $N = N(\Lambda)$ of $\Lambda = \mathrm{End}_R Q$ is nilpotent with index of nilpotency $\leqq n$. Using linkage we can improve upon this bound.

3.6. Proposition. *Let Q be a uniform quasi-injective α-smooth module with $\Lambda = \mathrm{End}_R Q$ and $N = \{f \in \Lambda \mid \mathrm{Ker} f \leqq_e Q\}$. Let D denote the minimal α-coprimitive ideal associated with $E(Q)$. Then the index of nilpotency of N is $\leqq$ the number of layers of Q to which D is linked.*

Proof. The proof is by induction on the number n of layers of Q to which D is linked. If $n = 1$, then D is linked only to the first layer of Q. As in 3.3 this implies that $J = 0$. Suppose D is linked to exactly n layers of Q where $n > 1$. Let $\mathrm{Sc}_t Q/\mathrm{Sc}_{t-1} Q$ be the last layer of Q to which D is linked. Then $\mathrm{Sc}_{t-1} Q$ is an α-smooth uniform quasi-injective module such that D is linked to exactly $n-1$ layers of $\mathrm{Sc}_{t-1} Q$. By induction if $N^* = N(\mathrm{End}_R \mathrm{Sc}_{t-1} Q)$, then $(N^*)^{n-1} = 0$.

Let $j_1, \cdots, j_n \in N$. Then $j_i \mid \mathrm{Sc}_{t-1} Q \in N^*$ which implies that $j_2 \cdots j_n(\mathrm{Sc}_{t-1} Q) = 0$. Thus $j_2 \cdots j_n(\mathrm{Sc}_t Q)$ is a homomorphic image of $\mathrm{Sc}_t Q/\mathrm{Sc}_{t-1} Q$ and hence by 2.5, $j_2 \cdots j_n(\mathrm{Sc}_t Q) \subset \mathrm{Sc}_1 Q$. By 2.5 then $j_1 j_2 \cdots j_n(\mathrm{Sc}_t Q) = 0$. Let k be the largest integer such that $j_1 j_2 \cdots j_n(\mathrm{Sc}_k Q) = 0$. If $\mathrm{Sc}_k Q \neq Q$, then $j_1 j_2 \cdots j_n$ induces a nonzero map $\mathrm{Sc}_{k+1} Q/\mathrm{Sc}_k Q \to Q$. By 3.1, D is linked to the $(k+1)$st layer of Q. However $k \geqq t$ and $\mathrm{Sc}_t Q/\mathrm{Sc}_{t-1} Q$ is the last layer of Q to which D is linked which is impossible. Thus $\mathrm{Sc}_k Q = Q$ and $j_1 j_2 \cdots j_n = 0$.

4. The localization of R with respect to $\mathcal{M}_\alpha$

In [7, 8.9], the following theorem appears providing a converse to the Teply–Miller Theorem [13]. We include a sketch of the proof for the convenience of the reader. Also see [14] and [15] by C. Nastasescu and [12] by G. Hansen.

4.1. Theorem. *Let M be a quasi-injective module over a ring A and let $\Lambda = \mathrm{End}_A M$, then the following are equivalent.*

(1) *M is a Δ-module.*

(2) *${}_\Lambda M$ is noetherian.*

(3) *${}_\Lambda M$ is artinian and noetherian.*

(4) *M satisfies the ascending chain conditions for right annihilators in A and the biendomorphism ring of M_A is semiprimary.*

Proof. The proof of (2)→(1) is direct and (3)→(2) is clear. To show (1)→(3), we assume first that M_A is injective. Let τ be the torsion theory cogenerated by M. By [17, p. 61], A satisfies the descending chain condition for τ-closed right ideals and hence by [13, 1.4], A satisfies the ascending chain condition for τ-closed right ideals. If $x_1, \cdots, x_n \in M$, then $(\Lambda x_1 + \cdots + \Lambda x_n)^{rl} = \Lambda x_1 + \cdots + \Lambda x_n$. Thus ${}_\Lambda M$ satisfies the ascending and descending chain condition for finitely generated submodules. Hence ${}_\Lambda M$ is noetherian, which in turn implies that ${}_\Lambda M$ is artinian.

If we now assume that M is a quasi-injective Δ-module, then M is finitely annihilated. Hence, M is a Δ-injective $A/\text{ann } M$-module. Since $\text{End}_A M = \text{End}_{A/\text{ann } M} M$, the result follows.

The implication (3)→(4) is a consequence of [9, p. 324] where it is shown that the endomorphism ring of a module of finite length is semiprimary.

Finally (4)→(1) is a consequence of the following result of [7, 8.9].

4.2. Proposition [C. Faith]. *Let M be a module over a semiprimary ring S and let $\theta : A \to S$ be a ring homomorphism such that $\theta(1_A) = 1_S$. If M as an A-module via θ satisfies the ascending chain condition for right annihilators, then M is a Δ-module.*

If τ is a torsion theory for a ring A, then as in [10] let $Q_\tau(M)$ denote the localization of M_A with respect to τ, let A_τ denote the localization of A with respect to τ and let $E_\tau(M)$ denote the τ-injective hull of M. From [10, p. 61] there exists a ring homomorphism $\hat{\tau} : A \to A_\tau$. From 1.4 and 1.5 of [15], we have

4.3. Proposition [C. Nastasescu]. *If E is a Δ-injective right A-module, then A_τ is a semiprimary ring, where τ is the torsion theory cogenerated by E. In addition E is a Δ-injective module over A_τ, and $A_\tau \cong \text{Biend}(E_R)$.*

If A is right noetherian, then every A_τ-module N is a Δ-module over A via the ring homomorphism $\hat{\tau} : A \to A_\tau$.

4.4. Corollary. *Let M be a right module over the right noetherian ring A, and let τ be a torsion theory. If R_τ is semiprimary, then $Q_\tau(M)$ is a Δ-module over R.*

If M_R is τ-torsionfree, then M is a Δ-module over R.

Proof. This follows from 4.2 and the fact that if M is torsionfree, then M embeds in $Q_\tau(M)$.

We now consider the ring R which is right noetherian and $|R| = \alpha$. If R is α-smooth, then $E(R)$ is α-smooth, and hence by 2.3, is a Δ-module. Thus from 4.3 we have

4.5. THEOREM. *If the ring R is α-smooth, then the complete ring of quotients $Q(R)$ is semiprimary.*

In [8], we show that there exists a 1–1 correspondence between the α-indecomposable injective modules, the α-primes and the minimal α-coprimitive ideals. In this correspondence, if I is an α-indecomposable injective, then I is associated with the α-prime $P = \operatorname{ass} I$ and the minimal α-coprimitive ideal $D = \operatorname{ann} SI$.

Let $J_1, \cdots, J_n$ denote a complete set of representatives of the isomorphism classes of α-indecomposable injectives. The topology $\mathcal{M}_\alpha = \{H < R \mid |R/H| < \alpha\}$ is determined by $J_1, \cdots, J_n$. It is also determined by $E(R/N)$ where N denotes the intersection of the α-primes, and by $E(R/K_\alpha(R))$ where $K_\alpha(R)$ denotes the intersection of the minimal α-coprimitive ideals.

4.6. PROPOSITION. *The ring $R_{\mathcal{M}_\alpha}$ is semiprimary and all modules over $R_{\mathcal{M}_\alpha}$ are Δ-modules over R.*

PROOF. Let $E = J_1 \oplus \cdots \oplus J_n$. By 2.3, E is a Δ-injective module. The result now follows from 4.3.

By applying the results of section 3, we can determine when $R_{\mathcal{M}_\alpha}$ is semisimple.

4.7. PROPOSITION. *The following are equivalent.*

(1) *$R_{\mathcal{M}_\alpha}$ is semisimple artinian.*

(2) *$SI = I$ for all α-indecomposable injective modules I.*

(3) *$K_\alpha(R) = T$, where T is the maximal right ideal with $|T| < \alpha$.*

PROOF. The equivalence of (2) and (3) is from [4, 4.7].

(3)→(1): If $\tau = \mathcal{M}_\alpha$, then $\tau(R) = T$. We now use [3, 6.5]. Let $E_R(R/\tau(R)) = I_1 \oplus \cdots \oplus I_n$ where I_i are α-indecomposable injectives for all i, where $1 \leqq i \leqq n$. Since $SI_i = I_i$ for all i, then $\mathrm{Sc}(I_1 \oplus \cdots \oplus I_n) = I_1 \oplus \cdots \oplus I_n$. Now $R/\tau(R) \leqq_e I_1 \oplus \cdots \oplus I_n$ and finitely generated submodules of $I_1 \oplus \cdots \oplus I_n$ are semicritical. Hence if $x \in I_1 \oplus \cdots \oplus I_n$, then $|xR + R/\tau(R)/R/\tau(R)| < \alpha$. Thus $E_\tau(R/\tau(R)) = I_1 \oplus \cdots \oplus I_n$. By 3.5, $\mathrm{End}(E_\tau(R/\tau(R)))$ is semisimple and equals $R_{\mathcal{M}_\alpha}$.

(1)→(3): Suppose $R_{\mathcal{M}_\alpha}$ is semisimple. Since each α-indecomposable injective module I is $\mathcal{M}_\alpha$-torsionfree, then I is an $R_{\mathcal{M}_\alpha}$-module. By [4, 3.1], SI is $\mathcal{M}_\alpha$-closed in I_α. Hence $Q_{\mathcal{M}_\alpha}(SI) = SI$ and therefore SI is also an $R_{\mathcal{M}_\alpha}$-module. Since $R_{\mathcal{M}_\alpha}$ is semisimple and I is indecomposable, necessarily $I = SI$.

4.8. Corollary. *If $R_{\mathcal{M}_\alpha}$ is semisimple, then $Q(R) = R_{\mathcal{M}_\alpha}$ if and only if $K_\alpha(R) = 0$.*

Proof. If $K_\alpha(R) = 0$, then by [4, 4.11], $Z(R) = 0$ and the set of large right ideals equals $\mathcal{M}_\alpha$. Thus $R_{\mathcal{M}_\alpha} = Q(R)$.

Conversely let $\tau = \mathcal{M}_\alpha$ and let $T = \tau(R)$. By [10, 6.11], then $R_{\mathcal{M}_\alpha} T = 0$. Hence $Q(R) \cdot T = 0$ which implies that $T = 0$. Since $R_{\mathcal{M}_\alpha}$ is semisimple $K_\alpha(R) = T$ by 4.7. Thus $K_\alpha(R) = 0$.

We close this section with an example to illustrate some of the results.

4.9. Example. Let

$$A = \begin{bmatrix} \mathbf{Z} & \mathbf{Z} & \mathbf{Z}_p \\ 0 & \mathbf{Z} & \mathbf{Z}_p \\ 0 & 0 & \mathbf{Z}_p \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \mathbf{Z} & \mathbf{Z} & \mathbf{Z}_p \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z}_p \\ 0 & 0 & \mathbf{Z}_p \end{bmatrix}.$$

Then A and B are right noetherian rings of Krull dimension 1. Now $A_{\mathcal{M}_1} = \left[\begin{smallmatrix} Q & Q \\ 0 & Q \end{smallmatrix}\right]$ which is semiprimary and $B_{\mathcal{M}_1} = \left[\begin{smallmatrix} Q & Q \\ Q & Q \end{smallmatrix}\right]$, which is semisimple.

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